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Solution of a Problem of Ulam

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In this paper we solve the following *Ulam problem*: "Give conditions in order for a linear mapping near an approximately linear mapping to exist" and establish results involving a product of powers of norms [S. M. Ulam, "A Collection of Mathematical Problems," Interscience, New York, 1961; "Problems in Modern Mathematics," Wiley, New York, 1964; "Sets, Numbers, and Universes," MIT Press, Cambridge, MA, 1974]. There has been much activity on a similar " ε-isometry" problem of Ulam [J. Gervirtz, *Proc. Amer. Math. Soc.* **89** (1983), 633–636; P. Gruber, *Trans. Amer. Math. Soc.* **245** (1978), 263–277; J. Lindenstrauss and A. Szankowski, "Non-linear Perturbations of Isometries," Colloquium in honor of Laurent Schwartz, Vol. I, Palaiseau, 1985]. This work represents an improvement and generalization of the work of D. H. Hyers [*Proc. Nat. Acad. Sci USA* **27** (1941), 222–224].

THEOREM. *Let X be a normed linear space with norm $\|\cdot\|_1$ and let Y be a Banach space with norm $\|\cdot\|_2$. Assume in addition that $f\colon X \to Y$ is a mapping such that $f(t\cdot x)$ is continuous in t for each fixed x. If there exist $a, b, 0 \leqslant a+b<1$, and $c_2 \geqslant 0$ such that*

$$\|f(x+y)-[f(x)+f(y)]\|_2 \leqslant c_2 \cdot \|x\|_1^a \cdot \|y\|_1^b \tag{1}$$

for all $x, y \in X$, then there exists a unique linear mapping $L\colon X \to Y$ such that

$$\|f(x)-L(x)\|_2 \leqslant c \cdot \|x\|_1^{a+b} \tag{2}$$

for all $x \in X$, where $c = c_2/(2-2^{a+b})$.

If one takes $a=b=0$ in this theorem and follows our proof, one obtains an additive functional L such that $\|f(x)-L(x)\|_2 \leqslant c_2$, for all x in X. This is Hyer's result [3].

Proof of Existence. Inequality (1) and $y=x$ imply

$$\|f(2x)-2f(x)\|_2 \leqslant c_2 \cdot \|x\|_1^{a+b},$$

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or

$$\|f(2x)/2 - f(x)\|_2 \leqslant c_2 \cdot \|x\|_1^{a+b}/2. \tag{3}$$

More generally, the following lemma holds.

LEMMA 1. *In the space X,*

$$\|f(2^n x)/2^n - f(x)\|_2 \leqslant c_2 \cdot \sum_{i=0}^{n-1} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b} \tag{4}$$

for some $c_2 \geqslant 0$ *and for any integer* n.

To prove Lemma 1 we proceed by induction on n.

For $n = 1$, the result is obvious from (3). We assume then that (4) holds for $n = k$ and prove that (4) is true for $n = k + 1$. Indeed, from (4) and $n = k$ and $2 \cdot x = z$ we find:

$$\|f(2^k z)/2^k - f(z)\|_2 \leqslant c_2 \sum_{i=0}^{k-1} 2^{i \cdot (a+b-1)-1} \cdot \|z\|_1^{a+b},$$

or

$$\|f(2^{k+1} \cdot x)/2^{k+1} - f(2x)/2\|_2 \leqslant c_2 \cdot \sum_{i=0}^{k-1} 2^{(i+1) \cdot (a+b-1)-1} \cdot \|x\|_1^{a+b},$$

or

$$\|f(2^{k+1} \cdot x)/2^{k+1} - f(2x)/2\|_2 \leqslant c_2 \sum_{i=1}^{k} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b}. \tag{5}$$

Therefore from (3) and (4) we get

$$\begin{aligned} &\|f(2^{k+1}x)/2^{k+1} - f(x)\|_2 \\ &\quad \leqslant \|f(2^{k+1} \cdot x)/2^{k+1} - f(2x)/2\|_2 + \|f(2x)/2 - f(x)\|_2 \\ &\quad \leqslant c_2 \cdot \sum_{i=1}^{k} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b} + c_2 \cdot \|x\|_1^{a+b} \cdot 2^{-1} \\ &\quad = c_2 \cdot \sum_{i=0}^{k} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b}, \end{aligned}$$

or (4) holds for $n = k + 1$, or

$$\|f(2^{k+1} \cdot x)/2^{k+1} - f(x)\|_2 \leqslant c_2 \sum_{i=0}^{k} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b}. \tag{6}$$

But

$$\sum_{i=0}^{n-1} 2^{i(a+b-1)} < \sum_{i=0}^{\infty} 2^{i(a+b-1)} = \frac{1}{1-2^{a+b-1}} = c_0. \tag{7}$$

Set

$$c = c_0 \cdot c_2/2. \tag{7$'$}$$

It is clear that (3) and (6) yield (4), completing the proof of Lemma 1.

Then Lemma 1, (7), and (7)′ imply

$$\|f(2^n \cdot x)/2^n - f(x)\|_2 \leqslant c \cdot \|x\|_1^{a+b} \tag{8}$$

for any $x \in X$, any positive integer n, and some $c_2 \geqslant 0$.

LEMMA 2. *The sequence* $\{f(2^n \cdot x)/2^n\}$ *converges.*

We first use (8) and the completeness of Y to prove that the sequence $\{f(2^n \cdot x)/2^n\}$ is a Cauchy sequence. In fact, if $i > j > 0$, then

$$\|f(2^i \cdot x)/2^i - f(2^j \cdot x)/2^j\|_2 = 2^{-j} \cdot \|f(2^i \cdot x)/2^{i-j} - f(2^j \cdot x)\|_2 \tag{9}$$

and if we set $2^j \cdot x = h$ in (9) and employ (8) we get

$$\begin{aligned}&\|f(2^i \cdot x)/2^i - f(2^j \cdot x)/2^j\|_2 \\ &\quad = 2^{-j} \cdot \|f(2^i x)/2^{i-j} - f(h)\|_2 < 2^{j \cdot (a+b-1)} \cdot c \cdot \|x\|_1^{a+b}\end{aligned}$$

or

$$\lim_{j \to \infty} \|f(2^i x)/2^i - f(2^j \cdot x)/2^j\|_2 = 0 \tag{10}$$

because $a, b \colon 0 \leqslant a + b < 1$.

It is obvious now from (10) and the completeness of Y that the sequence $\{f(2^n \cdot x)/2^n\}$ converges and therefore the proof of Lemma 2 is complete.

We set

$$L(x) = \lim_{n \to \infty} \frac{f(2^n \cdot x)}{2^n}. \tag{11}$$

It is clear from (1) and (11) that

$$\|f(2^n \cdot x + 2^n \cdot y) - [f(2^n \cdot x) + f(2^n \cdot y)]\|_2 \leqslant c_2 \cdot \|2^n \cdot x\|_1^a \cdot \|2^n \cdot y\|_1^b,$$

or

$$\begin{aligned}&2^{-n} \cdot \|f(2^n \cdot x + 2^n \cdot y) - [f(2^n \cdot x) + f(2^n \cdot y)]\|_2 \\ &\quad \leqslant c_2 \cdot 2^{(a+b-1)n} \cdot \|x\|_1^a \cdot \|y\|_1^b,\end{aligned}$$

or

$$\| \lim_{n\to\infty} [f(2^n \cdot (x+y))/2^n] - \lim_{n\to\infty} [f(2^n \cdot x)/2^n] - \lim_{n\to\infty} [f(2^n \cdot y)/2^n] \|_2 = 0,$$

or

$$\|L(x+y) - L(x) - L(y)\|_2 = 0 \qquad \text{for any} \quad x,\, y \in X,$$

or

$$L(x+y) = L(x) + L(y) \qquad \text{for any} \quad x,\, y \in X. \tag{12}$$

From (12) we get

$$L(q \cdot x) = q \cdot L(x) \tag{13}$$

for any $q \in Q$, where Q is the set of rationals.

LEMMA 3. *Let Y^+ be the space of continuous linear functionals and consider the mapping*

$$T\colon t \to g(L(t \cdot x)), \qquad \text{or} \qquad T\colon \mathbb{R} \to \mathbb{R} \tag{14}$$

such that

$$T(t) = g(L(t \cdot x)), \tag{15}$$

where $g \in Y^+$, $t \in \mathbb{R}$, and $x \in X$, $x := \text{fixed}$. Then T is a continuous mapping.

To prove Lemma 3 we proceed as follows: Let

$$T_n(t) = g\left(\frac{f(2^n \cdot x \cdot t)}{2^n}\right) \tag{16}$$

such that

$$T(t) = \lim_{n\to\infty} T_n(t), \tag{17}$$

where $x \in X$, $x := \text{fixed}$ and $t \in \mathbb{R}$, $g \in Y^+$.

Then $T_n(t)$ are continuous and therefore T is measurable as the pointwise limit of continuous mappings T_n. Moreover, T is a homomorphism with respect to addition "+," that is,

$$T(x+y) = T(x) + T(y) \tag{18}$$

for any x, $y \in \mathbb{R}$. It is clear now that (18) and the measurability of T imply that T is a continuous mapping and thus the proof of Lemma 3 is complete.

Then Lemma 3 and the fact that Y^+ separates points of Y yield the *linearity* of L. However, if we take limits on both sides of (8) as $n \to \infty$ we obtain (2). Therefore, we have proved the *existence* of a linear mapping $L: X \to Y$ which also satisfies (2).

Uniqueness. It remains to show the uniqueness part of our theorem.

Let M be a linear mapping $M: X \to Y$, such that

$$\|f(x) - M(x)\|_2 \leqslant c' \cdot \|x\|_1^{a'+b'}, \qquad c' \geqslant 0, \tag{19}$$

for any $x \in X$ where $a', b': 0 \leqslant a' + b' < 1$ and c' is a constant. If there exists a linear mapping $L: X \to Y$ such that (2) *holds, then*

$$L(x) = M(x) \tag{20}$$

for any $x \in X$.

To prove (20) we must prove the following

LEMMA 4. *If* (2) *and* (19) *hold, then*

$$\|L(x) - M(x)\|_2 \leqslant m^{a+b-1} \cdot c \cdot \|x\|_1^{a+b} + m^{a'+b'-1} \cdot c' \cdot \|x\|_1^{a'+b'} \tag{21}$$

for any $x \in X$.

The required result (21) follows immediately if we use inequalities (2) and (19), the linearity of L and M, as well as the triangle inequality. In fact,

$$L(x) = \frac{L(m \cdot x)}{m}, \qquad M(x) = \frac{M(m \cdot x)}{m},$$

$\|L(m \cdot x) - M(m \cdot x)\|_2 \leqslant L(m \cdot x) - f(m \cdot x)\|_2 + \|M(m \cdot x) - f(m \cdot x)\|_2$. Then if we apply (2) and (19) we obtain inequality (21) and the proof of Lemma 4 is complete.

It is clear now that (21) implies $\lim_{m \to \infty} \|L(x) - M(x)\|_2 = 0$ for any $x \in X$, completing the proof of (20). Thus the uniqueness part of our Theorem is complete, as well.

Remark. A Banach space Y is said to have the *approximation property* if for any compact set $K \subset Y$ and any $\varepsilon > 0$, there exists $P \in L(Y, Y)$ depending on K and ε, with finite-dimensional range such that

$$\|P(x) - x\| \leqslant \varepsilon$$

for any $x \in K$.

The approximation problem states: Is every compact operator T in $L(X, Y)$ a limit in the norm of operators with finite dimensional range?

The approximation problem has *a negative solution* in Banach spaces (which are not Hilbert spaces) and was solved in the negative by Per Enflo (1973) via an example of a separable reflexive Banach space that does not have the approximation property.

Query. What is the situation in the above theorem in case $a+b=1$?

References

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